ECONOMETRIC INSTITUTE

COMPLETE SOLUTION OF A THUE INEQUALITY

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REPORT 9561/B

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December 15, 1995

Abstract

We determine the complete set of rational integers x, y that satisfy the Thue inequality $\left|x^3+x^2y-2xy^2-y^3\right| \leq 10^6$.

1 Introduction

A few years ago A. Pethö [P] described a method to determine the 'small' solutions of a Thue inequality in a very efficient way. As an example he determined the solutions of

$$\left|x^3+x^2y-2xy^2-y^3\right| \leq 200$$

with $|y| \leq 10^{500}$ (which is considered to be 'small'). His method is based on the observation that for such a solution x, y (apart maybe from a few easily found very small solutions) the quotient $\frac{x}{y}$ is a convergent from the continued fraction expansion of a root of the polynomial associated to the Thue inequality (in the case of Pethö's example this polynomial is t^3+t^2-2t-1). In fact, the corresponding partial quotient in this continued fraction expansion must be extremely large compared to $|y|$, and from explicit computations one can simply see that this does not happen in the given range for $|y|$.

Recently, the explicit upper bounds for the solutions of Thue inequalities, that can be derived from the theory of linear forms in logarithms of algebraic numbers, have been sharpened considerably, a.o. by A. Baker and G. Wüstholz [BW], whose results we will use below. This might lead one to find out whether the continued fraction method of Pethö would be able in reasonable time to reach all the way up to the upper bound, and thus find all the solutions. More specifically, Pethö asked (his question reached me via H.J.J. te Riele) for a complete solution of the Thue inequality

$$\left|x^3+x^2y-2xy^2-y^3\right| \leq 10^6 \tag{1}$$

*The author is grateful to A. Pethö and H.J.J. te Riele for suggesting the problem, and for discussions and encouragement.

in $x, y \in \mathbb{Z}$, this time without an a priori bound on $|y|$. Finding all the solutions to (1) would be a considerable extension of the results in Pethö's paper [P].

For inequality (1), we will derive below an explicit upper bound for $|y|$, but unfortunately this bound is very large (initially we find $\exp(6.76717 \times 10^{15})$). Although the type of results on which this bound is based are being sharpened over and over again, it is not to be expected that these methods of the theory of linear forms in logarithms will be able to provide bounds that are essentially much smaller. Maybe one could have some hope that for the solutions of (1) the upper bound $\exp(10^{10})$ for $|y|$ could be reached in the near future, but probably not much better than that.

If we would try the idea of Pethö's paper [P] to treat (1), we would end up with the following. When $|y|$ is large enough (below we find $|y| > 1550750$), then $\frac{x}{y}$ is a best approximation of one of the roots of $t^3 + t^2 - 2t - 1$, and thus in principle the solutions in certain prescribed ranges can be found as convergents from the simple continued fraction expansion of these roots. Moreover, it follows that the partial quotient corresponding to this convergent must be large compared to the denominator $|y|$ (larger than $c|y|$ for some constant c). Numerators and denominators of convergents tend to grow exponentially, and a few million of the partial quotients are not too hard to compute. This would roughly cover the range for $|y|$ up to $\exp(10^6)$ or maybe to $\exp(10^7)$.

However, the upper bound to be reached is so large that one would need a number of partial quotients which is of the order of magnitude of 10^{15}, which is way too much. Even if we adopt the bound $\exp(10^{10})$ for $|y|$, we would need a number of partial quotients which is of the order of magnitude of 10^{10}, and this too seems totally out of reach by the present state of hardware and software.

Nevertheless, in this note we will completely solve the Thue inequality (1), under the additional condition that x and y are coprime. The solutions with $\gcd(x, y) > 1$ can then easily be recovered. Our approach is different from that of Pethö in [P]. We will view the Thue inequality (1) as a set of one million Thue equations, viz.

$$x^3 + x^2y - 2xy^2 - y^3 = k, \tag{2}$$

for $k \in \{1, 2, \ldots, 10^6\}$, and solve each of these Thue equations by the nowadays routine method outlined in [TW1] (this method has been incorporated in the KANT software). Note that by changing signs of x and y it is not necessary to consider negative k, and obviously there are no solutions with $k = 0$ other than the trivial solution $x = y = 0$. Further, if (x, y) is a solution of (2), then so are $(-x - y, x)$ and $(y, -x - y)$, with the same k. This implies that we may restrict ourselves to solutions with $xy \geq 0$.

In comparison with Pethö's method, our method has the disadvantage that we have to repeat a solution procedure for very many values of k, whereas in the continued fraction method the parameter k almost playes no rôle at all. But we will see that we can exclude over 92% of the values for k at a very early stage, and that we have to perform a highly uniformized computational procedure for the remaining cases. Thus this procedure leads in only a few hours of computer time to a considerable reduction of the upper bound for $|y|$ (in fact, to 1.77311×10^{11}, as we will find below). To find the solutions below this reduced bound can then best be done by the continued fraction method of Pethö [P]. This time no more than 25 convergents are needed. Thus we find that with the present state of theory and technology, our method for solving the Thue inequality (1) seems more efficient than the continued fraction method.

We follow essentially the line of reasoning outlined in [TW1], adapted to deal with the special situation of many Thue equations that differ only in the constant term k, but not in the binary form. We will give full details of our proof below. Here's our main result.

Theorem 1 *The Thue inequality* (1) *has exactly* 8430 *solutions* $x, y \in \mathbb{Z}$ *satisfying* $\gcd(x, y) = 1$ *and* $xy \geq 0$.
In Table 1 *below*[1] *we list those solutions satisfying* $\max\{|x|, |y|, |x+y|\} > 10^4$.

We do not list all the solutions of (1) because that would take too much space. But note that it is easy to compute the solutions with $\gcd(x, y) > 1$ or $xy < 0$ or $\max\{|x|, |y|, |x+y|\} \leq 10^4$, and thus find the complete list of solutions from the statement of Theorem 1.

We also list in Table 2 below the solutions for which there are at least three other solutions with the same value of $x^3 + x^2y - 2xy^2 - y^3$.

2 Computations in a cubic field

Put $\theta = 2\cos\frac{2}{7}\pi = 1.24697\ldots$, and $\mathbb{K} = \mathbb{Q}(\theta)$. Then $\theta^3 + \theta^2 - 2\theta - 1 = 0$, and the following data are well known: the discriminant of $\mathbb{K}$ is 49, the field is Galois, and the automorphism $\sigma : \theta \longrightarrow \theta^2 - 2$ generates the Galois group, a set of fundamental units is given by $\{\theta, \sigma(\theta)\}$, a basis for the ring of integers $\mathcal{O}_{\mathbb{K}}$ is given by $\{1, \theta, \theta^2\}$, and the class group is trivial.

Equation (2) is thus equivalent to

$$x - y\theta = \alpha\theta^m\sigma(\theta)^n, \tag{3}$$

where α runs through a complete set of generators of integral ideals of $\mathbb{K}$ with norm satisfying $1 \leq N\alpha \leq 10^6$, and $m, n \in \mathbb{Z}$ are unknowns. Because of the coprimeness condition, the Prime Ideal Removing Lemma (Lemma 1 of [TW2]) tells us that we do not have to consider primes p that remain prime in $\mathcal{O}_{\mathbb{K}}$, and that the only ramifying prime, 7, can occur in k with exponent at most 1.

So our first task is to compute all splitting primes below 10^6, and find generators of prime ideals lying above them. Then we have to form all possible products of these ideals such that the norm of the product is $\leq 10^6$. Pari-1.38 on a 486/33 PC took about 4 hours to complete this job. The number of splitting primes is 26218, and the number of α's is 79689 up to conjugates. Of them, 69722 are not divisible by the prime ideal above the ramifying prime. Counting with conjugates, the number of α's is $3 \cdot 79689 - 2 = 239065$ (where we subtract 2 because $\alpha = 1$ equals its own conjugates). We won't bore the reader with listing them all.

[1]The tables can be found at the end of this paper.

3 Estimating linear forms in logarithms

The so-called Siegel identity, relating the three conjugates of $x - y\theta$, reads

$$\left(\sigma(\theta) - \sigma^2(\theta)\right)(x - y\theta) + \left(\sigma^2(\theta) - \theta\right)(x - y\sigma(\theta)) + (\theta - \sigma(\theta))\left(x - y\sigma^2(\theta)\right) = 0.$$

We use

$$\begin{aligned} \sigma(\theta) - \sigma^2(\theta) &= \left(1 - 2\theta - \theta^2\right)\sigma(\theta), \\ \sigma^2(\theta) - \theta &= \left(1 - 2\theta - \theta^2\right), \\ \theta - \sigma(\theta) &= \left(1 - 2\theta - \theta^2\right)\theta\sigma(\theta) \end{aligned}$$

and (3) to obtain from the Siegel Identity the so-called unit equation

$$\frac{-\sigma^2(\alpha)}{\sigma(\alpha)}\theta^{-m+2n+1}\sigma(\theta)^{-2m+n+1} - 1 = \frac{x - y\theta}{x - y\sigma(\theta)}. \tag{4}$$

We will show that the right hand side of this equation is extremely small.

Note that σ acts on the solutions as follows:

$$\begin{aligned} x - y\theta \quad \longrightarrow \quad x - y\sigma(\theta) &= \sigma(\theta)((-x - y) - x\theta) \\ \longrightarrow \quad x - y\sigma^2(\theta) &= \theta^{-1}(y - (-x - y)\theta), \end{aligned}$$

hence if (x, y) is a solution of (2), then so are $(-x - y, x)$ and $(y, -x - y)$. Of these three there is exactly one with $xy \geq 0$. So from now on, on changing the sign of k if necessary, we may assume that $x \geq 0$ and $y \geq 0$. Notice that without loss of generality we may assume a lower bound for y, as long as it is explicit and small enough to admit enumeration techniques for finding the solutions with y below this lower bound.

Our first lemma shows that $x - y\theta$ is in absolute value the smallest of the three conjugates.

Lemma 1 *If $y \geq 141$, then*

$$|x - y\theta| < |x - y\sigma(\theta)|, \quad |x - y\theta| < \left|x - y\sigma^2(\theta)\right|.$$

Proof. By $\sigma(\theta) = -0.44504\ldots$ we have $|x - y\sigma(\theta)| = x + y|\sigma(\theta)| \geq 0.44504\,|y|$, and by $\sigma^2(\theta) = -1.80193\ldots$ we have $|x - y\sigma^2(\theta)| = x + y\,|\sigma^2(\theta)| \geq 1.80193\,|y|$. Hence

$$|x - y\theta| = \frac{|k|}{|x - y\sigma(\theta)|\,|x - y\sigma^2(\theta)|} \leq \frac{10^6}{0.44504 \cdot 1.80193\, y^2},$$

and if $y \geq 141$ this is less than $0.44504\, y$. □

The second lemma shows that in fact $x - y\theta$ is extremely near to 0, whereas its two conjugates are far away from 0.

Lemma 2 *If $y \geq 1000$ then*

$$
\begin{array}{lrl}
(5) & 0.19370\, y^{-2} < |x - y\theta| & < 775375\, y^{-2} \\
(6) & 0.84601\, y < |x - y\sigma(\theta)| & < 1.69281\, y \\
(7) & 1.52445\, y < \left|x - y\sigma^2(\theta)\right| & < 3.04970\, y
\end{array}
$$

Proof. The left hand side of (6) follows by

$$y|\theta - \sigma(\theta)| \leq |x - y\theta| + |x - y\sigma(\theta)|$$

and Lemma 1, and similarly we find the left hand side of (7). Then, as in the proof of Lemma 1 we obtain the right hand side of (5):

$$10^6 \geq |k| = |x - y\theta||x - y\sigma(\theta)| \left|x - y\sigma^2(\theta)\right| > 0.84601 \cdot 1.52445\, y^2.$$

Further, the right hand side of (6) follows by $y \geq 1000$ from

$$|x - y\sigma(\theta)| \leq |x - y\sigma(\theta)| + y|\theta - \sigma(\theta)| < \left(\frac{775375}{y^3} + 1.69203\right) y,$$

and similarly we find the right hand side of (7). Finally, the left hand side of (5) follows by

$$1 \leq |k| = |x - y\theta||x - y\sigma(\theta)| \left|x - y\sigma^2(\theta)\right| < 1.69281 \cdot 3.04970\, y^2.$$

□

Lemma 2 shows that the absolute value of the right hand side of the unit equation (4) is extremely small indeed. Now we put

$$p = -m + 2n + 1, \quad q = -2m + n + 1,$$

and we introduce the linear form in logarithms

$$\Lambda = \log \left|\frac{\sigma^2(\alpha)}{\sigma(\alpha)}\right| + p \log|\theta| + q \log|\sigma(\theta)|.$$

Then the left hand side of the unit equation (4) can be written as $e^{\pm\Lambda} - 1$, and we obtain that Λ is extremely near to 0. Indeed, we have the following lemma.

Lemma 3 *If $y \geq 1000$ then*

$$|\Lambda| < 916931\, y^{-3}. \qquad (8)$$

Proof. By (4), (5) and (6) we have

$$\left|\frac{-\sigma^2(\alpha)}{\sigma(\alpha)}\theta^p \sigma(\theta)^q - 1\right| < \frac{775375}{0.84601} y^{-3} < 916509\, y^{-3}.$$

If $y \geq 1000$ then $916509 y^{-3} \leq 0.000916509$. If $|e^t - 1| < 0.000916509$ then $|t| < 1.00046|e^t - 1|$, so with $t = \Lambda$ we obtain the result. □

4 Derivation of the initial upper bounds

The next step is to establish relations between y and the coefficients p, q of the linear form Λ.

Lemma 4 *If* $y \geq 1000$ *then*

$$\begin{array}{rcccl} -27.75774 + 4.62220 \log y & < & p & < & 4.58732 + 4.62221 \log y, \\ -12.47042 + 1.26019 \log y & < & q & < & 6.15354 + 1.26020 \log y. \end{array}$$

Proof. Equation (3) implies

$$\begin{pmatrix} \log|x - y\sigma(\theta)| \\ \log|x - y\sigma^2(\theta)| \end{pmatrix} = \begin{pmatrix} \log|\sigma(\alpha)| \\ \log|\sigma^2(\alpha)| \end{pmatrix} + \begin{pmatrix} \log|\sigma(\theta)| & \log|\sigma^2(\theta)| \\ \log|\sigma^2(\theta)| & \log|\theta| \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix},$$

hence

$$\begin{aligned} \begin{pmatrix} p \\ q \end{pmatrix} &= \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2.66140\ldots & 1.96079\ldots \\ 1.96079\ldots & -0.70060\ldots \end{pmatrix} \begin{pmatrix} \log|x - y\sigma(\theta)| - \log|\sigma(\alpha)| \\ \log|x - y\sigma^2(\theta)| - \log|\sigma^2(\alpha)| \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned}$$

If $y \geq 1000$ then (6) and (7) imply that $\log|x - y\sigma(\theta)| > 0$ and $\log|x - y\sigma^2(\theta)| > 0$, and further we computed that for all our α's

$$|\alpha|, |\sigma(\alpha)|, \left|\sigma^2(\alpha)\right| \in [1, 546.876).$$

Now the result follows easily by (6) and (7). □

We even can get rid of the q, and we find that p is nonnegative.

Lemma 5 *If* $y \geq 24020$ *then* $\max\{|p|, |q|\} = p$.

Proof. At once from Lemma 4. □

The theory of linear forms in logarithms of algebraic numbers provides a lower bound for $|\Lambda|$. We can choose here from a variety of results. The result of A. Baker and G. Wüstholz [BW] that we use has a particularly easy statement, is explicit and very general, and gives very good bounds.

Lemma 6 *If* $y \geq 24020$ *then*

$$|\Lambda| > \exp\left(-5.34505 \times 10^{14} \log p\right).$$

Proof. Apply Lemma 5 and the main result from [BW], with $n = 3$, $d = 3$, and note that $h'\left(\frac{\sigma^2(\alpha)}{\sigma(\alpha)}\right) < 5.83331$, and $h'(\theta) = h'(\sigma(\theta)) = \frac{1}{3}$. □

For $|\Lambda|$ we now have an upper bound by Lemma 3, and a lower bound by Lemma 6. They can be glued together by Lemmas 4 and 5, to obtain absolute upper bounds for all the parameters.

Theorem 2

$$|p| < 3.12792 \times 10^{16}, \quad |q| < 8.52799 \times 10^{15}, \quad y < \exp\left(6.76717 \times 10^{15}\right).$$

Proof. If $y < 1000$ then all solutions are easily found, and the inequalities follow by enumeration. If $1000 \le y < 24020$ then they follow from Lemma 4. If $y \ge 24020$ then on the one hand we have Lemma 6, and on the other hand Lemmas 3, 4 and 5 yield

$$|\Lambda| < 916931 \exp\left(-3\,\frac{p - 4.58732}{4.62221}\right).$$

The upper bound for $|p|$ follows from comparing these upper and lower bounds for $|\Lambda|$, and then Lemma 4 gives the upper bounds for y and $|q|$. □

5 Reduction of the upper bounds

At this point the problem has become a finite one. But the upper bounds of Theorem 2 still are very large. Fortunately a method is known to reduce the bounds considerably. We will describe this method, originally due to A. Baker and H. Davenport [BD], in terms of lattices (cf. [TW1]).

Theorem 3

$$|p| \le 124, \quad |q| \le 38, \quad y < 1.77311 \times 10^{11}.$$

Proof. Consider the lattice $\Gamma = \{\mathcal{A}\mathbf{x} | \mathbf{x} \in \mathbb{Z}^2\}$, with

$$\mathcal{A} = \begin{pmatrix} 1 & 0 \\ [10^{50}\log|\theta|] & [10^{50}\log|\sigma(\theta)|] \end{pmatrix},$$

and for each α the point

$$\mathbf{y}_\alpha = \begin{pmatrix} 0 \\ \left[10^{50}\log\left|\frac{\sigma^2(\alpha)}{\sigma(\alpha)}\right|\right] \end{pmatrix}.$$

Here $[\cdot]$ means rounding to an integer in some sense. Note that

$$\begin{aligned} [10^{50}\log|\theta|] &= 22072\,43102\,83032\,98072\,50391\,20829\,25175\,45473\,32140\,16160, \\ [10^{50}\log|\sigma(\theta)|] &= -80958\,69160\,44712\,71255\,88813\,84774\,86778\,49031\,89231\,26386. \end{aligned}$$

A crucial point here is that the lattice Γ does not depend on α. Hence we have only one lattice for all our 239065 different α's.

For any solution p, q of inequality (8) in Lemma 3, we consider the point

$$\begin{pmatrix} p \\ \lambda \end{pmatrix} = \mathcal{A}\begin{pmatrix} p \\ q \end{pmatrix} - \mathbf{y}_\alpha.$$

Then

$$\lambda = \left[10^{50} \log \left| \frac{\sigma^2(\alpha)}{\sigma(\alpha)} \right|\right] + p \left[10^{50} \log |\theta|\right] + q \left[10^{50} \log |\sigma(\theta)|\right],$$

and we see that λ is relatively close to $10^{50}\Lambda$, namely, the rounding to an integer and Theorem 2 give us

$$|\lambda - 10^{50}\Lambda| \leq 1 + |p| + |q| < 3.98072 \times 10^{16}. \tag{9}$$

Our aim now is to find the distance between the point $\mathbf{y}_\alpha$ and the nearest lattice point, because that will give us a lower bound for the length of the vector $\begin{pmatrix} p \\ \lambda \end{pmatrix}$. But then we first have to exclude a few exceptional cases, in which the point $\mathbf{y}_\alpha$ is extremely close to a lattice point. These are just the cases of $\alpha = 1$ and the three α's with norm equal to 7, the ramifying prime.

If $\alpha = 1$ then $\mathbf{y}_\alpha = \mathbf{0}$, and hence $\mathbf{y}_\alpha = \mathcal{A}\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \Gamma$. This corresponds to the solution $p = 0, q = 0$.

If $\alpha = -1 + 2\theta + \theta^2$, then $\mathbf{y}_\alpha = \mathcal{A}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which is almost in Γ. This case corresponds to $p = 1, q = 0$.

If $\alpha = \sigma(-1+2\theta+\theta^2) = -2-\theta+\theta^2$, then $\mathbf{y}_\alpha = \mathcal{A}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \Gamma$, and this corresponds to $p = 0, q = 1$.

If $\alpha = \sigma^2(-1 + 2\theta + \theta^2) = 3 - \theta - 2\theta^2$, then $\mathbf{y}_\alpha = \mathcal{A}\begin{pmatrix} -1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, which is almost in Γ. This case corresponds to $p = -1$, $q = -1$.

In all the other cases we will show that in fact the distance between the point $\mathbf{y}_\alpha$ and the nearest lattice point is of the size of $\sqrt{\det \Gamma} \approx 10^{25}$, as can generically be expected.

We computed a reduced basis of the lattice Γ. This basis turned out to be

$$\left\{ \begin{pmatrix} -86891\,95858\,43504\,98794\,47659 \\ 75369\,67934\,76147\,47683\,41168 \end{pmatrix}, \begin{pmatrix} 26640\,08026\,06666\,30299\,01361 \\ 70064\,19181\,46661\,70387\,49382 \end{pmatrix} \right\}.$$

This computation can be done in far less than 1 second on a PC. Now let

$$d(\Gamma, \mathbf{y}_\alpha) = \min_{\mathbf{x} \in \mathbb{Z}^2} |\mathcal{A}\mathbf{x} - \mathbf{y}_\alpha|,$$

where in the four exceptional cases the one particular point $\mathbf{x} = \begin{pmatrix} p \\ q \end{pmatrix}$ is excluded from the set over which we take the minimum. It took about 4 hours on a 486/33 PC to compute this number $d(\Gamma, \mathbf{y}_\alpha)$ for each of the α's, using the above reduced basis. Notice that for each α the points $\mathbf{y}_\alpha$ and $\mathbf{y}_{(-1+2\theta+\theta^2)\alpha}$ differ at most by 1 in each coordinate. Hence we only had $3 \cdot 69722 - 2 = 209164$ different cases to compute. They all satisfied

$$d(\Gamma, \mathbf{y}_\alpha) > 1.64991 \times 10^{22}.$$

It follows by Theorem 2 for all cases but the four exceptional ones, that

$$\left(1.64991 \times 10^{22}\right)^2 < d(\Gamma, \mathbf{y}_\alpha)^2 \leq p^2 + \lambda^2 < \left(3.12792 \times 10^{16}\right)^2 + \lambda^2,$$

whence

$$|\lambda| > 1.64490 \times 10^{22}.$$

Then by (9)

$$|\Lambda| \geq 10^{-50}\left(|\lambda| - \left|\lambda - 10^{50}\Lambda\right|\right) > 1.64489 \times 10^{-28}.$$

Finally, (8) now yields the reduced upper bound for y, and Lemma 4 the bounds for $|p|$ and $|q|$. □

6 Finishing the proof

We now finish the proof of Theorem 1.

Proof. We only have to find the solutions below the bound of Theorem 3. As remarked above, if $y > 1550750$, then $\frac{x}{y}$ is a convergent from the continued fraction expansion $[a_0, a_1, a_2, \ldots]$ of θ. We need only 25 convergents, because the denominator of the 25th convergent is $4.39288 \ldots \times 10^{12}$, which is already larger than the upper bound 1.77311×10^{11} of Theorem 3. All the convergents $\frac{x}{y}$ with $1550750 < y < 1.77311 \times 10^{11}$ satisfy $|x^3 + x^2y - 2xy^2 - y^3| > 10^6$. We give the first 30 partial quotients and convergents of θ in Table 3 at the end of this paper.

Hence $y \leq 1550750$, and this bound is small enough to admit enumeration of the remaining cases. Note that for given y, inequality (5) does not leave much room for x. So this enumeration costs only a few minutes on a PC. In this way we found the result announced in the theorem, and computed the Tables below. □

As a corollary we have that the total number of solutions satisfying $\gcd(x, y) = 1$ is $3 \cdot 8430 = 25290$. Further, it is easy to show that the total number of solutions satisfying $xy \geq 0$, but not necessarily x, y coprime, is 13860, and that the total number of solutions without any condition thus is $3 \cdot 13860 = 41580$.

Finally, as an illustration, we remark that for the largest solution $x = 715371, y = 573683$ we have the following data:
$x - y\theta = 0.00000055234\ldots$ (in fact, $\frac{715371}{573683}$ is the 9th convergent from the continued fraction expansion of θ), $x - y\sigma(\theta) = 970683.9\ldots$, $x - y\sigma^2(\theta) = 1749112.0\ldots$, $\alpha = 115 - 37\theta - 91\theta^2$, $x - y\theta = \alpha\theta^7\sigma(\theta)^{25}$, $p = 44$, $q = 12$, and $\Lambda = 2.532\ldots \times 10^{-13}$, which is exceptionally small indeed for a linear form with coefficients p, q of this size.

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Tables

x	y	$-x-y$	k
−5675	−4551	10226	446251
5781	4636	−10417	283529
−5882	−4717	10599	320333
−5983	−4798	10781	966547
−6089	−4883	10972	172817
−6296	−5049	11345	2521
−6397	−5130	11527	728657
6710	5381	−12091	411139
−6811	−5462	12273	399463
6917	5547	−12464	656867
7124	5713	−12837	930103
7225	5794	−13019	30491
−7326	−5875	13201	920899
−7533	−6041	13574	712781
7639	6126	−13765	570661
−7947	−6373	14320	212563
8154	6539	−14693	81901
8568	6871	−15439	765449
−8669	−6952	15621	549779
9083	7284	−16367	155063
−9391	−7531	16922	993511
9497	7616	−17113	998857
−9598	−7697	17295	614257
−9805	−7863	17668	199037
10012	8029	−18041	253331
10941	8774	−19715	380059
−11456	−9187	20643	705083

x	y	$-x-y$	k
−11663	−9353	21016	105407
11870	9519	−21389	538601
−12385	−9932	22317	724723
12799	10264	−23063	732311
13521	10843	−24364	95159
13728	11009	−24737	964543
−14243	−11422	25665	695687
−15172	−12167	27339	640303
15379	12333	−27712	429493
−17030	−13657	30687	431033
17237	13823	−31060	924427
−17959	−14402	32361	270439
19817	15892	−35709	179437
20746	16637	−37383	475427
−24255	−19451	43706	530711
26113	20941	−47054	268199
−30551	−24500	55051	901349
32409	25990	−58399	346319
38705	31039	−69744	398671
45001	36088	−81089	410129
51297	41137	−92434	365567
57593	46186	−103779	249859
63889	51235	−115124	47879
−70185	−56284	126469	255499
−76481	−61333	137814	675401
−134074	−107519	241593	721519
−325741	−261224	586965	525091
715371	573683	−1289054	937789

Table 1: The solutions of the Thue inequality (1) with $\gcd(x,y)=1$, $xy \geq 0$, $k = x^3+x^2y-2xy^2-y^3 > 0$, and $\max\{|x|,|y|,|x+y|\} > 10^4$. With (x,y) also $(-x-y,x)$ and $(y,-x-y)$ are solutions of (1), with the same k.

k	(x,y)						
181	(−4, −5),	(14, 11),	(−16, −13),	(−101, −81)			
559	(8, 1),	(9, 5),	(−4, −7),	(−31, −25),	(929, 745)		
1189	(−1, −10),	(−5, −9),	(−19, −16),	(−56, −45)			
1247	(22, 17),	(−13, −12),	(−23, −19),	(−308, −247)			
1261	(−9, −10),	(11, 5),	(34, 27),	(−61, −49),	(−76, −61)		
2059	(−4, −11),	(13, 6),	(33, 26),	(−722, −579)			
2899	(14, 5),	(−19, −17),	(−22, −19),	(126, 101)			
8497	(−5, −18),	(20, 7),	(63, 50),	(227, 182)			
17753	(34, 23),	(−57, −47),	(−70, −57),	(1343, 1077)			
26767	(32, 17),	(−23, −27),	(257, 206),	(−4438, −3559)			
38429	(36, 19),	(−20, −29),	(−74, −61),	(211, 169)			
47879	(−9, −32),	(36, 1),	(−14, −31),	(−54, −47),	(133, 106),	(−369, −296),	(63889, 51235)
60229	(38, 11),	(−25, −34),	(−94, −77),	(−179, −144)			
67159	(−1, −40),	(−29, −36),	(−55, −49),	(−919, −737)			
76609	(41, 8),	(50, 31),	(−49, −46),	(−54, −49),	(−319, −256)		
94627	(45, 17),	(72, 53),	(−40, −43),	(−142, −115)			
127387	(−19, −43),	(−28, −43),	(−43, −47),	(−153, −124)			
151003	(−67, −61),	(148, 117),	(−187, −151),	(−682, −547)			
163241	(−1, −54),	(−41, −49),	(−71, −64),	(−350, −281)			
164891	(61, 35),	(89, 66),	(−51, −53),	(−793, −636)			
212563	(173, 137),	(−175, −142),	(−232, −187),	(−662, −531),	(−7947, −6373)		
220597	(87, 62),	(−53, −57),	(123, 95),	(227, 181)			
241613	(−2, −61),	(99, 73),	(−32, −53),	(−223, −180)			
298831	(89, 61),	(−71, −69),	(−113, −96),	(1065, 854)			
380393	(−41, −62),	(−81, −77),	(103, 73),	(135, 103)			
406783	(73, 27),	(99, 68),	(−32, −63),	(119, 88),	(640, 513)		
424801	(−9, −70),	(−31, −64),	(−86, −81),	(−145, −121)			
474643	(−27, −67),	(−46, −67),	(−67, −73),	(248, 197)			
664651	(85, 9),	(−11, −81),	(−25, −76),	(517, 414),	(−241, −196),	(−1939, −1555)	
729611	(91, 40),	(117, 79),	(−121, −108),	(−779, −625)			

Table 2: The solutions of the Thue equations (2) with $\gcd(x,y) = 1$, $xy \geq 0$, for those $k \in \{1, 2, \ldots, 10^6\}$ for which at least four such solutions exist.

n	a_n	x	y	k
0	1	1	1	−1
1	4	5	4	1
2	20	101	81	−181
3	2	207	166	197
4	3	722	579	−2059
5	1	929	745	559
6	6	6296	5049	−2521
7	10	63889	51235	47879
8	5	325741	261224	−525091
9	2	715371	573683	937789
10	2	1756483	1408590	−3960503
11	1	2471854	1982273	3353183
12	2	6700191	5373136	−8354641
13	2	15872236	12728545	44488151
14	1	22572427	18101681	−4846603
15	18	422175922	338558803	957329477
16	1	444748349	356660484	−819086359
17	1	866924271	695219287	924626177
18	3	3045521162	2442318345	−4174958017
19	2	6957966595	5579855977	15978254957
20	1	10003487757	8022174322	−12016580953
21	2	26964942109	21624204621	65348466331
22	1	36968429866	29646378943	−41283590671
23	2	100901801841	80916962507	299931185527
24	1	137870231707	110563341450	−14221068707
25	39	5477840838414	4392887279057	8443033023151
26	2	11093551908535	8896337899564	−22818072480589
27	1	16571392746949	13289225178621	26351476483891
28	1	27664944655484	22185563078185	−68462532216161
29	1	44236337402433	35474788256806	12732446673479
30	13	602737330887113	483357810416663	−1768139351083097

Table 3: The first 30 partial quotients a_n and convergents $\frac{x}{y}$ of the continued fraction expansion of $\theta = 2\cos\frac{2}{7}\pi$, and the corresponding $k = x^3 + x^2y - 2xy^2 - y^3$.